Real Talk about Real Numbers

Rodica D. Costin [*]

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1 Sets

In mathematics, any collection or system of objects is called a **set**.

The objects a set A is made of are called **elements of the set** A. If x is one of these objects we say that x is an element of A (or, that x *belongs to* A), and denote $x \in A$.

If y *is not* an element of A we write $y \notin A$ (read: y does not belong to A).

Example of sets:

the collection of chairs in room CC226, an interval (a, b) (which is the collection of all real numbers x with $a < x < b$), the set consisting of the numbers 2, 3, 4, 5 and 6: $\{2, 3, 4, 5, 6\}$ (when enumerating the elements of a set, you do so between $\{$ and $\}$).

The empty set is the set with no elements; it is denoted $\emptyset$.

1.1 Subsets

Given two sets A, B we say that A **is included in** B (or that A is **a subset** of B) and denote $A \subset B$, if any element of A is also an element of B:

$$A \subset B \quad \text{if and only if} \qquad x \in A \;\; \text{implies} \;\; x \in B$$

If $A \subset B$ we can also say that B **includes** A, and denote $B \supset A$.

[*]Department of Mathematics, Ohio-State University

Examples:
1) The set of natural numbers is a subset of the real numbers.
2) $(1,3) \subset [1,3) \subset [1,4)$
3) The empty set is included in any set: $\emptyset \subset A$.
4) $A \subset A$.

1.2 Operations with sets

Union $A \cup B$ is the set which collects all elements of A and B:

$$x \in A \cup B \text{ if and only if } x \in A \text{ or } x \in B$$

Examples:
1) $\{0,1,2,3,4\} \cup \{3,4,5,6\} = \{1,2,3,4,5,6\}$.
2) The domain of the function $f(x) = 1/x$ is the set $(-\infty, 0) \cup (0, +\infty)$.
3) $\emptyset \cup A = A$
Note: $A \cup B = B \cup A$.

Intersection $A \cap B$ is the set which collects all elements common to A and B:

$$x \in A \cap B \text{ if and only if } x \in A \text{ and } x \in B$$

Examples:
1) $\{0,1,2,3,4\} \cap \{3,4,5,6\} = \{3,4\}$
2) $[0,2] \cap [1,2) = [1,2)$
3) $\emptyset \cap A = \emptyset$
Note: $A \cap B = B \cap A$.

Difference $A \setminus B$ is the set which collects all elements of A which do not belong to B:

$$x \in A \setminus B \text{ if and only if } x \in A \text{ and } x \notin B$$

Examples:
1) $\{0,1,2,3,4\} \setminus \{3,4,5,6\} = \{0,1,2\}$.
2) $[0,10] \setminus [1,2] = [0,1) \cup (2,10]$
3) $A \setminus A = \emptyset, \qquad A \setminus \emptyset = A$.

2 Real numbers: decimal representation

Natural numbers: 0, 1, 2, 3, 4, 5, 6,...

Integer numbers: $... - 6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, ...$

Rational numbers: these are ratios of two integers: m/n where m, n are integers and $n \neq 0$.

Decimal representation of rational numbers: doing long division it turns out that eventually decimal repeat periodically.

Examples:

$$\frac{1}{3} = 0.333333... = 0.(3)$$

$$\frac{139}{330} = 0.421212121... = 0.4(21)$$

$$\frac{5}{4} = 1.25 = 1.2500000... = 1.25$$

Conversion from decimal representation to fraction:

$$1.25 = 1 + \frac{2}{10} + \frac{5}{100}$$

$$0.421 = 0 + \frac{4}{10} + \frac{2}{100} + \frac{1}{1000}$$

$$0.4(21) = 0 + \frac{4}{10} + \frac{21}{990}$$

A wrinkle: repeating 9's is the same as a 0, for example

$$0.499999... = 0.4(9) = 0.5 \quad , \quad 0.999999... = 0.(9) = 1.0 \quad \text{and} \quad 2.34(9) = 2.35$$

Irrational numbers: integral part followed by an infinity of decimals, not ending by a repeating sequence. For example

$$3.141592653589793238...$$

These are the first digits of a special number which we call π. The number π is defined through its properties (π is the ratio between the circumference of any circle and its diameter), and not through its decimals. As another example, $\sqrt{2} = 1.4142135623730...$ is that number whose square is 2; this number turns out to be irrational.

When we do calculations for practical applications, we do use decimals, but we stop after a finite number of decimals: in fact we take an approximation by a rational number.

3 Bounds for sets of numbers

3.1 Upper bounds of a set

Consider S a set of real numbers.

S is called **bounded above** if there is a number M so that any $x \in S$ is less than, or equal to, M: $x \leq M$. The number M is called *an upper bound* for the set S.

Note that if M is an upper bound for S then any bigger number is also an upper bound.

Not all sets have an upper bound. For example, the set of natural numbers does not.

A number B is called **the least upper bound** *(or* **supremum***) of the set S if:*

1) B is an upper bound: any $x \in S$ satisfies $x \leq B$, and

2) B is the smallest *upper bound. In other words, any smaller number is not an upper bound:*

$$\textit{if} \;\; t < B \;\; \textit{then there is } x \in S \;\; \textit{with} \;\; t < x$$

Notation:

$$B = \sup S = \sup_{x \in S} x$$

Upper bounds of S may, or may not belong to S.

For example, the interval $(-2, 3)$ is bounded above by 100, 15, 4, 3.55, 3. In fact 3 is its least upper bound.

The interval $(-2, 3]$ also has 3 as its least upper bound.

When the supremum of S is a number that belongs to S then it is also called the **maximum** of S.

Examples:

1) The interval $(-2, 3)$ has supremum equal to 3 and no maximum; $(-2, 3]$ has supremum, and maximum, equal to 3.

2) The function $f(x) = x^2$ with domain $[0, 4)$ has a supremum (equals 4^2), but not a maximum. The function $g(x) = x^2$ with domain $[0, 4]$ has (not only a supremum, but also) a maximum; it equals $g(4) = 4^2$.

The interval $(-2, +\infty)$ is not bounded above.

If the set S is not bounded above (also called *unbounded above*) we write (conventionally)

$$\sup S = +\infty$$

3.2 Bounded sets do have a least upper bound.

This is a fundamental property of real numbers, as it allows us to talk about limits.

Theorem *Any nonempty set of real numbers which is* bounded above *has a supremum.*

Proof.

We need a good notation for a real number given by its decimal representation. A real number has the form

$$a = a_0.a_1a_2a_3a_4... \qquad \text{where } \; a_0 \text{ is an integer and } a_1, a_2, a_3, ... \in \{0, 1, 2, ...9\}$$

To eliminate ambiguity in defining real numbers by their decimal representation, let us decide that if the sequence of decimals ends up with nines: $a = a_0.a_1a_2...a_n9999...$ (where $a_n < 9$) then we choose this number's decimal representation as $a = a_0.a_1a_2...(a_n + 1)0000....$ (For example, instead of 0.4999999.. we write 0.5.)

Let S be a nonempty set of real numbers, bounded above.

Let us construct the least upper bound of S.

Consider first all the approximations by integers of the numbers a of S: if $a = a_0.a_1a_2...$ collect the a_0's. This is a collection of integer numbers. It is bounded above (by assumption). Then there is a largest one among them, call it B_0.

Next collect only the numbers in S which begin with B_0. (There are some!) Call their collection S_0.

Any number in $S \setminus S_0$ (number of S not in S_0) is smaller than any number in S_0.

Look at the first decimal a_1 of the numbers in S_0. Let B_1 be the largest among them. Let S_1 be the set of all numbers in S_0 whose first decimal is B_1.

Note that the numbers in S_1 begin with $B_0.B_1$

Also note that any number in $S \setminus S_1$ is smaller than any number in S_1.

Next look at the second decimal of the numbers in S_1. Find the largest, B_2 etc.

Repeating the procedure we construct a sequence of smaller and smaller sets $S_0, S_1, S_2, ...S_n, ...$

$$S \supset S_0 \supset S_1 \supset S_2 \supset ... \supset S_n \supset ...$$

Note that every set S_n contains al least one element (it is not empty).

At each step n we have constructed the set S_n of numbers of S which start with $B_0.B_1B_2...B_n$; the rest of the decimals can be anything. Also all numbers in $S \setminus S_n$ are smaller than all numbers of S_n. (The construction is by induction!)

We end up with the number $B = B_0.B_1B_2...B_nB_{n+1}....$

We need to show that B is the least upper bound.

To show it is an upper bound, let $a \in S$. If $a_0 < B_0$ then $a < B$. Otherwise $a_0 = B_0$ and we go on to compare the first decimals. Either $a_1 < B_1$ therefore $a < B$ or, otherwise, $a_1 = B_1$. Etc. So either $a < B$ or $a = B$. So B is an upper bound.

To show it is the least (upper bound), take any smaller number $t < B$. Then t differs from B at some first decimal, say at the nth decimal: $t = B_0.B_1B_2...B_{n-1}t_nt_{n+1}...$ and $t_n < B_n$. But then t is not in S_n and S_n contains numbers bigger than t. QED

3.3 Lower bounds

By exchanging "less than" $<$ with "greater than" $>$ throughout the section §3.1 we can similarly talk about lower bounds.

Here it is.

S is called *bounded below* if there is a number m so that any $x \in S$ is bigger than, or equal to m: $x \geq m$. The number m is called *a lower bound* for the set S.

Note that if m is a lower bound for S then any smaller number is also a lower bound.

A number b is called **the greatest lower bound** *(or* **infimum***) of the set S if:*

1) b is a lower bound: any $x \in S$ satisfies $x \geq b$, and

2) b is the greatest *lower bound. In other words, any greater number is not a lower bound:*

$$\text{if } b < t \text{ then there is } x \in S \text{ with } x < t$$

Notation:

$$b = \inf S = \inf_{x \in S} x$$

Greatest lower bounds of S may, or may not belong to S. For example, the interval $(-2, 3)$ is bounded below by -100, -15, -4, -2. In fact -2 is its infimum (greatest lower bound). The interval $[-2, 3)$ also has -2 as its infimum.

When the infimum of S belongs to S then it is called the **minimum** of S.

The interval $(-\infty, -2)$ is not bounded below.

If the set S is not bounded below we write (conventionally)

$$\inf S = -\infty$$

Theorem ***Any nonempty set of real numbers which is bounded below has an infimum.***

Proof.

No, we need not repeat the proof of §3.2. We do as follows.

Let S be a nonempty set which is bounded below. Construct the set T which contains all the opposites $-a$ of the numbers a of S:

$$T = \{-a\,;\ \text{where } a \in S\}$$

The set T is nonempty and is bounded above. By the Theorem of §3.2, T has a least upper bound, call it B. Then its opposite, $-B$, is the greatest lower bound for S. Q.E.D.

3.4 Bounded sets

A set which is bounded above and bounded below is called **bounded**.

So if S is a bounded set then there are two numbers, m and M so that $m \leq x \leq M$ for any $x \in S$. It sometimes convenient to lower m and/or increase M (if need be) and write $|x| < C$ for all $x \in S$.

A set which is not bounded is called *unbounded*.

For example the interval $(-2, 3)$ is bounded.

Examples of unbounded sets: $(-2, +\infty), (-\infty, 3)$, the set of all real numbers $(-\infty, +\infty)$, the set of all natural numbers.

4 Properties of real numbers

The set of real numbers has the following properties **I-V**:

I. Order properties

I.1 For any two real numbers a, b one, and only one of the following hold

$$a < b \quad \text{or} \quad b < a \quad \text{or} \quad a = b$$

I.2 Transitivity:

$$\text{If} \;\; a < c \quad \text{and} \quad c < b \quad \text{then} \quad a < b$$

I.3 Between any two numbers there is another number:

$$\text{If} \;\; a < b \quad \text{then there is} \;\; c \;\; \text{so that} \quad a < c < b$$

II. Addition

There is an operation between any two real numbers a, b, called addition, denoted $a + b$ so that:

II.1 $(a + b) + c = a + (b + c)$ (associativity)
II.2 $a + 0 = 0 + a = a$ (0 is a neutral element)
II.3 Any number a has an opposite, denoted $-a$, so that $a + (-a) = 0$.
II.4 $a + b = b + a$ (commutativity)
II.5 If $a < b$ than $a + c < b + c$.

III. Multiplication

There is an operation between any two real numbers a, b, called multiplication, denoted ab (or $a \cdot b$) so that:

III.1 $(ab)c = a(bc)$ (associativity)
III.2 $a \cdot 1 = 1 \cdot a = a$ (1 is a neutral element)
III.3 Any number $a \neq 0$ has an inverse, denoted $\frac{1}{a}$, so that $a \cdot \frac{1}{a} = 1$.
III.4 $ab = ba$ (commutativity)
III.5 $a(b + c) = ab + ac$ (distributivity)
III.5 If $a < b$ and $c > 0$ then $ac < bc$.

IV. Archimedean property

Given any number a there is a larger *natural* number n: $a < n$.

V. Existence of the least upper bound

Any nonempty set of real numbers which is bounded above has a supremum.

5 What are the Real numbers?

In practice we do not use the whole infinite sequence of decimals of an irrational number. What we do use are the properties of the given number.

Some of the general properties of real numbers were listed in §4. There are more, of course, but they can *all* be deduced from the listed five.

The modern approach is to define the set of real numbers through its properties:

Definition *A set with properties* **I-V** *is called the set of real numbers.*

This is an axiomatic definition: properties **I-V** are taken to be axioms - statements considered to be true. All other properties of real numbers will be deduced from these five, using logic.

When an axiom system is established there are two major questions:

1) Are there enough axioms to match our intuition on the concept we want to define?

In our case if we omit axiom **V**, the first four are also satisfied by the rational numbers!

(Note: by axiom **V** the real numbers are a completion of the rationals.)

2) Are they consistent? Is there a set for which the specified axioms are true?

Well, we do have our model with decimal representation of real numbers. They satisfy **I-III** by the way operations are defined, **IV** is very easy to show. Only **V** needed a proof.

6 Sequences

A sequence is an enumeration of numbers: $x_1, x_2, x_3, ...x_n, ...$ The order matters, so we can not call this a set. In fact, this is a function defined on the natural numbers: to each natural number n we associate a number x_n. (We chose here to start enumerating with 1 rather than with 0.)

Usual notation: $\{x_n\}$, or $\{x_n\}_n$ if there is a doubt what the index of enumaration is; n is a dummy variable.

Examples:

1) $\{n^2\}$ is the sequence $1^2,\ 2^2,\ 3^2,\ 4^2,\ ...n^2, ...$

2) $\{a+nr\}_n$ is the sequence $a+r,\ a+2r,\ a+3r, ...a+nr, ..$ (an arithmetic progression ratio r).

It may be more convenient to consider the arithmetic progression starting with a:

$$a,\ a+r,\ a+2r,\ a+3r, ..a+nr, ...$$

and then the notation is $\{a+nr\}_{n\geq 0}$.

3) $\{br^n\}_{n\geq 0}$ is the sequence $b,\ br,\ br^2,\ ...br^n, ...$ (a geometric progression).

Definition of a convergent sequence

A sequence $\{x_n\}$ is said to have the limit L (or $\{x_n\}$ converges *to L) if for any $\epsilon > 0$ there is a natural number N so that*

$$|x_n - L| < \epsilon \quad \textit{for all} \;\; n > N$$

Note that the number N depends on ϵ (usually the smaller ϵ the larger N needs to be), so sometimes we write N_ϵ.

Notation:

$$\lim_{n\to\infty} x_n \;=\; L \qquad \text{or simply} \qquad x_n \to L$$

Exercise: Show that the limit of the sequence $\{\frac{1}{n}\}$ is zero.

Examples:

$$\lim_{n\to\infty} \frac{2n+1}{n+2} = 2 \qquad , \qquad \lim_{n\to\infty} \frac{1}{n}\sin n = 0$$

$$\lim_{n\to\infty} \frac{1}{c^n} = 0 \;\text{ if } |c| > 1$$

$$\lim_{n\to\infty} r^n \ = 0 \ \text{ if } |r| < 1$$

The following two exercises are to be left for Wednesday after Exam II.

Exercise: Show that 0.999999... = 1.

Important Exercise: Denote by s_n the sum of the first n terms of a geometric progression. Find for which numbers b and r the sum s_n is convergent and find its limit in these cases.

All theorems we leaned about limits of functions are also true for limits of sequences.

You may have noted the similarity between the definition of limit for sequences and that of functions. In fact, the two are closely related. The following theorem shows that to check if a function has the limit L as $x \to a$ it is enough to check this statement on *all* sequences $x_n \to a$:

Theorem *A function $f(x)$ has the limit L as $x \to a$:*

$$\lim_{x\to a} f(x) = L$$

if and only if

for any sequence $\{x_n\}$ which converges to a the sequence $\{f(x_n)\}$ converges to L:

$$\textit{for any} \quad x_n \to a \quad \textit{we have} \quad f(x_n) \to L$$

The proof of this theorem is not included here.

Example: With this theorem it is very easy to show that the function $f(x) = \sin(\frac{1}{x})$ has no limit as $x \to 0$. How?

Monotone sequences

A sequence $\{x_n\}$ is called **increasing** if $x_n < x_{n+1}$ for all n.

Similarly, $\{x_n\}$ is called **decreasing** if $x_n > x_{n+1}$ for all n.

Theorem *Any sequence which is increasing (or decreasing) and is bounded has a limit.*

Can you guess what the limit is?

Exercise: Prove this theorem.

7 Uniform continuity

For many purposes, mere continuity of a function is not enough (continuity does not behave well when we take sequences of functions). A slightly more demanding notion works much better.

Definition of uniform continuity *A function f defined on an interval I is called* uniformly continuous *on I if:*
for any $\epsilon > 0$ there is $\delta > 0$ so that

$$\textit{for all } x_0, x_1 \in I \textit{ with } |x_1 - x_0| < \delta \textit{ we have } |f(x_1) - f(x_0)| < \epsilon \tag{1}$$

Remark 1: *a uniformly continuous function is, in particular, continuous.*

Indeed, $f(x)$ is continuous on I if it is continuous at all points in I, therefore if: for all $x_0 \in I$ we have $\lim_{x\to x_0} f(x) = f(x_0)$.
Now writing what limit means, we get: f is continuous on I if for all $x_0 \in I$ and all $\epsilon > 0$ there is $\delta > 0$ so that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

What is the difference between the above statement (of continuity on I) and the definition of uniform continuity on I? In the above statement the δ may depend on x_0, while in the case of uniform continuity it does not! In the case of a uniformly continuous function we can choose a δ which works for all the $x_0 \in I$. (But of course, δ will still depend on ϵ.)

There are many uniformy continuous functions:

Theorem 2 *A function continuous on a closed interval is uniformly continuous.*

Before taking a glimpse at the proof of this important theorem, let us see some examples, and how to recognize uniform continuity (when it is the case).

Example 1. The function $f(x) = 1/x$ is uniformly continuous on the interval $[1, 2]$. (Because it is continuous on a closed interval.)

Functions which are continuous on a non-closed interval may, or may not be uniformly continuous.

Example 2. The function $f(x) = 1/x$ is uniformly continuous on the interval $(1, 2)$. (Because it is continuous on the larger interval $[1, 2]$.)

Example 3. A function which is differentiable on (a, b), with derivative bounded on (a, b), is uniformly continuous on (a, b).

Indeed, consider as an example the function $f(x) = 1/x$. Let us show it is uniformly continuous on $(1, +\infty)$.

Let $\epsilon > 0$. We need to find δ so that (1) holds.

Let $x_0, x_1 \in (1, +\infty)$. There is a c between x_0 and x_1 such that

$$f(x_1) - f(x_0) \;=\; f'(c)\,(x_1 - x_0) \tag{2}$$

by the Mean Value Theorem (since $f(x)$ is continuous on $[x_0, x_1]$, and differentiable.)[1]

But $f'(x) = -1/x^2$ which is bounded on $(1, +\infty)$:

$$-1 < -1/x^2 < 0 \quad \text{for all } x \in (1, +\infty)$$

Therefore $-1 < f'(c) < 0$; or $|f'(c)| < 1$.

Now let us take absolute value on both sides of the equality (2):

$$|f(x_1) - f(x_0)| \;\le\; |f'(c)|\,|x_1 - x_0| \;<\; |x_1 - x_0|$$

It is clear now that the definition of uniform continuity we can take $\delta = \epsilon$. Q.E.D.

Example 4. The function $f(x) = x$ is uniformly continuous on $(-\infty, +\infty)$. Why?

Remark: If the derivative is not bounded the function may, or may not, be uniformly continuous.

Example 5. The function $f(x) = \sqrt{x}$ is uniformly continuous on $(0, 2)$ (why?) while its derivative is not bounded on $(0, 2)$ (why?).

Example 6. The function $f(x) = 1/x$ is continuous on the interval $(0, 2]$, but not uniformly. Let us show this.

To show that a function is *not* uniformly continuous, it is enough to find two sequences of points x_0 and x_1 so that $x_1 - x_0$ converge to 0, but $f(x_1) - f(x_0)$ does not. (Why?)

For Example 6 take $x_0 = \frac{1}{n}$ and $x_1 = \frac{1}{2n}$.

Example 7. The function $f(x) = x^2$ is uniformly continuous on $[0, 100]$ (why?), but not on $[0, +\infty)$ (why?).

[1] The number c depends on x_1 and on x_0.

What emerges from the above examples is the intuitive picture that a uniformly continuous function does not "stretch" distances too much.

Main ideas in the proof of Theorem 2

Let f be continuous on $[a, b]$. Then for any $x_0 \in I$ and any $\epsilon > 0$ there is $\delta = \delta_{x0,\epsilon} > 0$ so that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$ (see the explanations following Remark 1.)

We would like to take the number δ in the definition of uniform continuity to be the smallest among all the $\delta_{x0,\epsilon}$ (then the theorem would be proved). The problem is that there are infinitely many numbers $\delta_{x0,\epsilon}$ (they are indexed by all the points $x_0 \in I$), and their infimum may be zero.

Now comes the crucial fact that the interval is closed.

A fundamental property of closed intervals (which will not be proved here) states the following. Suppose a closed interval $[a, b]$ is covered by the union of infinitely many *open* intervals. Then finitely many among them are enough to still cover $[a, b]$.

In our case $[a, b]$ is included in the union of all the intervals $I_{x_0} = (x_0 - \delta_{x0,\epsilon}\ ,\ x_0 + \delta_{x0,\epsilon})$.

But a number n among them will still be enough to cover $[a, b]$: there is $x_{0,1}, ..., x_{0,n}$ so that the union $I_{x_{0,1}} \cup ... \cup I_{x_{0,n}}$includes $[a, b]$. Now take $\delta_\epsilon = \min\{\delta_{x_{0,1},\epsilon}, ..., \delta_{x_{0,1},\epsilon}\}$ and Theorem 2 follows.